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LETTER TO THE EDITOR

Stationary solutions of stochastic parabolic and hyperbolic sine–Gordon equations

S Albeverio†‡§, Z Haba‡§‖ and F Russo‖¶

† Mathematisches Institut, Ruhr-Universität, D-4630 Bochum 1, CERFIM (Locarno), Federal Republic of Germany
‡ SFB 237 Bochum-Essen-Düsseldorf, Federal Republic of Germany
§ BiBos, Universität Bielefeld, Federal Republic of Germany
‖ Institute of Theoretical Physics, University of Wroclaw, Poland
¶ Département de Mathématiques, Université de Provence, Marseille, France

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Abstract. We consider the damped sine–Gordon equation perturbed by (thermal) spacetime noise in the form it arises in the theory of the Josephson junction and charge density waves. We announce a rigorous proof that the coupling constant expansion of the solution of the initial value problem converges for small coupling. Taking the limit of the initial time $t_0 \to -\infty$ we obtain a stationary solution. We show that the stationary solution is localized both in time and space even if the solution at finite time is not.

The sine–Gordon equation with a random perturbation has generated a lot of interest recently as a successful description of solid state phenomena [1–5]. The equation discussed has the form

$$Q\partial_t\varphi - \Delta\varphi + \mu^2\partial_t^2\varphi + \lambda \sin \alpha\varphi + m^2\varphi = P\eta \tag{1}$$

where η is the spacetime white noise (with $t, t' \in \mathbb{R}$, $x, x' \in \mathbb{R}^d$)

$$\langle \eta(t,x)\eta(t',x')\rangle = \delta(t-t')\delta(x-x') \tag{2}$$

($\langle\ \rangle$ meaning expectation). $\mu^2, m^2 \geqslant 0$ and α, $\lambda \in \mathbb{R}$ are parameters, Q, P certain operators.

In the Josephson junction $d=1$ and $Q\partial_t\varphi$ is a dissipation assumed in [1] to be of the form $\Delta\partial_t\varphi$. The term $m^2\varphi$ is absent in [1] (it would violate the invariance $\varphi \to \varphi + (2\pi/\alpha)n$) but we keep it here as a regularization. In the theory of charge density waves [3]–[6] $\mu^2\partial_t^2$ is the Newton inertia [4] (μ is the mass of the charged particles). It is usually neglected as the experiments are in good agreement with the assumption $\mu = 0$[4], $Q =$ constant and $m = 0$ in accordance with the interpretation of φ as a phase. The operator P in (1) is usually taken to be equal to 1. However, we keep it different from 1 as a regularization of the noise and we shall discuss later whether we may let P converge to 1 or not. If Q or P are not constants the relativistic invariance is violated.

Our study of (1) is at an early stage, especially concerning such non-perturbative phenomena as turbulent conductivity in charge density waves [4] and soliton propagation in the Josephson junction. We give, however, a rigorous formulation of (1) and show the existence of an equilibrium state. We prove the existence of the solution of (1) as a stationary Markov process. We show that the correlation functions of the stationary solution are exponentially decaying both in time and space even if the solution at finite times has long-range correlations. This can be seen as a justification of the 1-site approximation used in [4].

We formulate (1) in the framework of a Hilbert space valued stochastic process (a forthcoming paper will discuss two-parameter random fields solving (1)).

Let $\vartheta = (\varphi/\pi)$ then (1) can be expressed in the form

$$\mathrm{d}\vartheta = \mathrm{i}A\vartheta\,\mathrm{d}t + I(\vartheta)\,\mathrm{d}t + P\,\mathrm{d}W \tag{3}$$

where

$$A = -\mathrm{i}\begin{pmatrix} -Q & \dfrac{1}{\mu^2} \\ \Delta - m^2 & 0 \end{pmatrix} \tag{4}$$

$$I = \begin{pmatrix} 0 \\ \lambda \sin \alpha\varphi \end{pmatrix} \qquad \frac{\mathrm{d}W}{\mathrm{d}t} = \begin{pmatrix} 0 \\ \eta \end{pmatrix}.$$

If $\mu = 0$ then we write (1) in the form

$$Q\,\mathrm{d}\varphi = (\Delta - m^2)\varphi\,\mathrm{d}t + \lambda \sin(\alpha\varphi)\,\mathrm{d}t + P\,\mathrm{d}W. \tag{5}$$

A is a self-adjoint operator in the Hilbert space of integrable functions with the inner product $(\vartheta, \vartheta') = (\omega\varphi, \omega\varphi') + (\pi, \pi')$ [7]. We transform (1) into an integral equation

$$\vartheta_t = \exp[\mathrm{i}A(t-t_0)]\vartheta_0 + \int_{t_0}^{t} \exp[\mathrm{i}A(t-\tau)]I(\vartheta_\tau)\,\mathrm{d}\tau + \int \exp[\mathrm{i}A(t-\tau)]\,P\,\mathrm{d}W \tag{6}$$

where

$$\exp[\mathrm{i}A(t-t_0)] = \begin{pmatrix} \exp[-Q(t-t_0)] & 0 \\ 0 & 1 \end{pmatrix} \exp[\mathrm{i}B(t-t_0)] \tag{7}$$

and

$$\exp(\mathrm{i}Bt) = \begin{pmatrix} \cos t\omega & \omega^{-1} \sin t\omega \\ -\omega \sin t\omega & \cos t\omega \end{pmatrix} \tag{8}$$

with $\omega = (-\Delta + m^2)^{1/2}$. The integral form of (5) reads (we set here $Q = 1$)

$$\varphi_t = \exp[(\Delta - m^2)(t-t_0)]\varphi_0 + \int_{t_0}^{t} \exp[(\Delta - m^2)(t-\tau)]\lambda \sin \alpha\varphi_\tau\,\mathrm{d}\tau + \int_{t_0}^{t} [\exp(t-\tau)(\Delta - m^2)]P\,\mathrm{d}W_\tau. \tag{9}$$

We shall investigate here only weak solutions of (6) and (9) in the form of a perturbation series in λ, i.e. we ask the question whether the perturbation series in powers of λ converges.

The first-order approximation to (6) is

$$\psi_t=\int_{t_0}^{t}\omega^{-1}\sin\omega(t-\tau)\exp[-Q(t-\tau)]P\,\mathrm{d}W_\tau \tag{10}$$

whereas the first-order approximation to (9) is

$$\tilde{\varphi}_t=\int_{\tau_0}^{t}\exp[(t-\tau)(\Delta-m^2)]P\,\mathrm{d}W_\tau. \tag{11}$$

Let us denote

$$M=\mathrm{i}\omega-Q/\mu.$$

In terms of M the correlation function of Ψ reads

$$\begin{aligned}E_{t_0}[\psi_t(x)\psi_{t'}(x')]=-\left(\frac{P}{2\omega}\right)^2\int_{t_0}^{t}\mathrm{d}\tau\{&\exp M(t+t'-2\tau)+\exp M^*(t+t'-2\tau)\\&-\exp(Mt-M\tau+M^*t'-M^*\tau)-\exp(Mt'-M\tau+M^*t-M^*\tau)\}.\end{aligned} \tag{12}$$

The expansion in λ of the solution of (3) is somewhat involved, if we would like to do it directly, because consecutive approximations enter the argument of $\sin\alpha\varphi$. A better way is supplied by the Cameron–Martin–Girsanov–Maruyama formula from the theory of stochastic equations [8].

The theorem says the following: Assume P is invertible with the inverse P^{-1}, let φ_t be the solution of (3) and ψ be the first-order approximation (10), then for any regular integrable F (depending on φ_τ, $\tau\leqslant t$)

$$\langle F(\varphi)\rangle=\langle\rho_{t_0}^{t}(\psi)F(\psi)\rangle \tag{13}$$

where

$$\rho_{t_0}^{t}(\psi)=\exp\left\{\lambda\int_{t_0}^{t}\sin\alpha\psi_\tau P^{-1}\,\mathrm{d}W_\tau-\frac{\lambda^2}{2}\int_{t_0}^{t}(P^{-1}\sin\alpha\psi_\tau)^2\,\mathrm{d}\tau\right\}. \tag{14}$$

In (13) and (14) ψ is a functional of W. So, the expectation value on the RHS is with respect to the Wiener measure. ρ^t fulfills the stochastic equation

$$\mathrm{d}\rho^t=\rho^t\lambda\sin\alpha\psi_\tau P^{-1}\,\mathrm{d}W_t. \tag{15}$$

Equation (15) can be solved by iteration, what leads to the expansion in λ of ρ and of the expectation values (13)

$$\rho=1+\lambda\int_{t_0}^{t}\sin\alpha\psi_\tau P^{-1}\,\mathrm{d}W_\tau+\lambda^2\int_{t_0}^{t}\sin\alpha\psi_\tau\left(\int_{t_0}^{\tau}\sin\alpha\psi_{\tau'}P^{-1}\,\mathrm{d}W_{\tau'}\right)P^{-1}\,\mathrm{d}W_\tau. \tag{16}$$

The same formula applies to the parabolic equation (9). So

$$\langle F(\varphi)\rangle=\langle\rho_{t_0}^{t}(\tilde{\varphi})F(\tilde{\varphi})\rangle. \tag{17}$$

The only difference in (13) and (17) is in the dependence of ψ and $\tilde{\varphi}$ on W (compare (10) and (11), respectively). This similarity permits their simultaneous treatment and has as a consequence some similarity of the behaviour of their solutions.

There are usually two problems intrinsically connected with stochastic differential equations in spacetime, which in the language of a field theorist can be called the infrared and ultraviolet problems (or large distance and short distance behaviour, respectively).

The first concerns the behaviour for large t and x; the second the behaviour for nearly coinciding arguments of time and space. This behaviour depends on the arbitrary operators (or numbers) P, Q, which we inserted into the equations and which must be discussed now.

First, we wish to consider the limit $t_0 \to -\infty$, which is necessary if we wish to derive a solution which does not depend on the initial condition. This is justified from the physical point of view if we are interested in phenomena in materials a long time after the prepartaion of the experiment. The correlations observed should not depend on the initial time the experiment starts.

Let us consider first the limit $t_0 \to -\infty$. We obtain from (12)

$$\lim_{t_0 \to -\infty} E_{t_0}[\psi_t(x)\psi_{t'}(x')] = \tfrac{1}{8} P\omega^{-2}P\{M^{-1}\exp(M|t-t'|) + M^{*-1}\exp(M^*|t-t'|)$$

$$-2(M+M^*)^{-1}(\exp(M^*|t-t'|) + \exp(M|t'-t|))\}(x, x'). \tag{18}$$

The RHS of (18) is the kernel of the operator

$$\tfrac{1}{4}\exp\left(-\frac{Q}{\mu^2}|t-t'|\right)\left(\frac{PQ^{-1}P}{\dfrac{Q^2}{\mu^2}+\omega^2}\cos(\omega(t-t')) + \frac{P\omega^{-1}P}{\omega^2+\dfrac{Q^2}{\mu^2}}\sin(\omega|t-t'|)\right). \tag{19}$$

For $\tilde{\varphi}$ (equation (11)) we obtain

$$\lim_{t_0 \to -\infty} E_{t_0}[\tilde{\varphi}_t(x)\tilde{\varphi}_{t'}(x')] = [\exp -(m^2-\Delta)|t-t'|]P^2(m^2-\Delta)^{-1}(x, x'). \tag{20}$$

We can see in (19) that the dissipation increases the correlation length. Namely, $\omega^2 \to \omega^2 + Q^2/\mu^2$ in the denominator of (19). As a result

$$E_{-\infty}[\psi_t(x)\psi_{t'}(x')] \leq \text{constant} \exp -\frac{Q}{\mu}(|t-t'| + |x-x|)$$

for large times or distances, uniformly in m. This is true only for the stationary solution. It can be seen from (12) that $E_{t_0}[\psi_t(x)\psi_{t'}(x')]$ has a slow (power-like) decay if $m=0$.

From (19) and (20) we can see whether we can go on with the expansion in λ in order to get a non-trivial (weak) solution of (1). First of all we need the correlation functions of $\sin \alpha\varphi$ to be integrable functions (after possible renormalizations). We have

$$\langle \exp(\mathrm{i}\alpha\tilde{\varphi}_t(x)) \exp(-\mathrm{i}\alpha\tilde{\varphi}_{t'}(x'))\rangle = \exp\left[-\frac{\alpha^2}{2}\langle(\tilde{\varphi}_t(x) - \tilde{\varphi}_{t'}(x'))^2\rangle\right].$$

Assuming that we remove $\langle\tilde{\varphi}_L^2\rangle$ by 'normal or Wick ordering' (see the discussion preceding (26)) the requirement of integrability in (16) and (17) gives rise to a restriction on the allowed singularity ($x, x' \in \mathbb{R}^d$)

$$\frac{\alpha^2}{2}\langle\tilde{\varphi}_t(x)\tilde{\varphi}_{t'}(x')\rangle \langle (|d \ln|x-x'| + |\ln|t-t'||) \tag{21}$$

in the parabolic case $\mu=0$.

The hyperbolic case $\mu\neq 0$ is more involved because the singularities may lie on the light cone (this can be prevented if the dissipation is strong enough to effectively prohibit wave propagation.) The condition

$$\frac{\alpha^2}{2}\langle\psi_t(x)\psi_{t'}(x')\rangle\langle|\ln|x-x'|| + |\ln|t-t'|| \tag{22}$$

is sufficient for integrability of the correlation functions of $\sin\alpha\varphi_t$ in the first-order approximation.

From (20) we now get (p^2 is the symbol of $-\Delta$, i.e. the representation of $-\Delta$ in Fourier transformed space)

$$P^2(p^2)p^{-2} \leqslant \text{constant}\, p^{-d} \tag{23}$$

for the parabolic case.

The hyperbolic case (discussed for $Q=0$, $P=1$, $d=1$ in [9, 10]) is again more involved, because of the interplay between x and t singularities. In order to ensure integrability at $t=t'$ we need

$$\frac{P^2(p^2)Q^{-1}(p^2)}{p^2+Q^2(p^2)/\mu^2} \leqslant \text{constant}\, p^{-d}$$

and (24)

$$\frac{P^2(p^2)}{p^2+Q^2(p^2)/\mu^2} \leqslant \text{constant}\, p^{-d}.$$

If $Q^2(p^2)$ does not grow faster than p^2 for large p then the singularity in time is of the same form as the singularity in space. If Q grows faster than p^2 then the singularity in time can be weaker than in space. The singularity of the correlation function $\langle\psi_t\psi_{t'}\rangle$ is no longer on the light cone. This corresponds to the case of strong damping making the wave equation model similar to the parabolic case. In fact, in such a case we may neglect ω, i.e. $\cos\omega(t-t')\approx 1$, $\sin\omega(t-t')\approx 0$ and the two-point function (19) becomes similar to the parabolic one (20) depending on the relation between P and Q. Concerning the parabolic case for $d=1$ or the linear case for $d\geqslant 1$ [11, 12].

Through the Girsanov formulae (13) and (14) the weak solution of the stochastic equations can be reduced to the convergence problem of the expansion (16) and the study of the properties of the resulting perturbation series in λ.

First if P and Q are of such a form that ψ (respectively $\tilde{\varphi}$) correlations are continuous functions (corresponding to the case that the LHS of (23) and (24) decays faster than the RHS) then the series (16) converges due to the estimate $\langle W^{2n}\rangle\approx n!$ (because the time-ordered integral of $2n$ variables gives $(2n!)^{-1}$).

Moreover if the covariances $\langle\psi_t\psi_{t'}\rangle$ and $\langle\tilde{\varphi}_t\tilde{\varphi}_{t'}\rangle$ decay exponentially in time then the limit $t_0\rightarrow-\infty$ can be taken term by term. We can conclude that the solutions of the stochastic equations have the same decay properties as their first-order approximations (in particular we get for the complete solutions the exponential decay of correlations in space and time if the first-order term has this property).

We come finally to the borderline case of logarithmic singularity corresponding to the case where the LHS in (23) and (24) behaves exactly as the RHS, i.e. as p^{-d}. This case is particularly interesting for the parabolic equation in $d=2$ (in which case we can take $P=1$).

The limit $t_0 \to -\infty$ defines a stationary measure which is of the heuristic form

$$d\mu_0(\varphi) = \exp\left[-\int \mathscr{L}(\varphi(x))\, dx\right] d\varphi \tag{25}$$

where $\mathscr{L}$ is a local function

$$\mathscr{L}(\varphi) = \tfrac{1}{2}\varphi(-\Delta)\varphi + \lambda \cos \alpha\varphi$$

For the locality the choice $P=1$ is crucial. For $d=2$, $\alpha^2 < 4\pi$ this can be given a meaning as a free field measure perturbed by a trigonometric interaction see [13–15]. The locality ensures the Markov property of μ_0 [15].

Stochastic dynamics leading to the invariant measure of the Markov form (25) has been studied rigorously in [16–19] but mainly for polynomial and exponential interactions. The $\cos \alpha\varphi$ case describes the equilibrium classical mechanics of a Bose gas of charged particles. The process φ_t could be associated with a non-equilibrium state of a boson gas [20].

In order to make sense out of $\sin \alpha\psi$ we need to normal order it, i.e.

$$\sin \alpha\psi \to \langle \sin \alpha\psi \rangle^{-1} \sin \alpha\psi \equiv :\sin \alpha\psi:$$

This corresponds to the renormalization of the coupling constant

$$\lambda \to \tilde{\lambda} \langle \sin \alpha\psi \rangle^{-1}.$$

After the renormalization the nth-order term in the expansion in powers of λ of (17) for the two-point function reads (we write the formula for the parabolic case with $P=1$)

$$\frac{d^n}{d\tilde{\lambda}^n} \langle :\sin \alpha\varphi_{t_1}(x_1)::\sin \alpha\varphi_{t_2}(x_2):\rangle_{|\tilde{\lambda}=0}$$

$$= \langle :\sin \alpha\tilde{\varphi}_{t_1}(x_1)::\sin \alpha\tilde{\varphi}_{t_2}(x_2):$$

$$\int_{t_0}^{t} :\sin \alpha\tilde{\varphi}_{\tau_n}(y_n):\, dW_{\tau_n}(y_n) \int \ldots \int \sin \alpha\tilde{\varphi}_{\tau_1}(y_1)\, dW_{\tau_1}(y_1)\rangle. \tag{26}$$

In order to get rid of the stochastic integrals in (26) we may use repeatedly the integration by parts formula (familiar from quantum field theory [21–23], and Malliavin calculus)

$$\langle F(\tilde{\varphi})\dot{W}_t(x)\rangle = \int_{\mathbb{R}^d}\int_{t_0}^{t} \frac{\delta F(\varphi)}{\delta\varphi_\tau(u)}\frac{\delta\varphi_\tau(u)}{\delta\dot{W}_t(x)}\,\mathrm{d}u\,\mathrm{d}\tau.$$

Denote

$$\frac{\delta\varphi_\tau(u)}{\delta\dot{W}_t(x)} = \sigma(\tau, u; t, x)$$

then, computing the expectation values in (26) we get

$$\left\langle \int \prod_{l,j} \sigma(s_l, u_l, \tau_j, y_j)\exp\left(\frac{\alpha^2}{2}\sum_{i,k} \pm\langle\tilde{\varphi}_{\tau_k}(y_k)\tilde{\varphi}_{\tau_i}(y_i)\rangle\right)\right\rangle \tag{27}$$

($\pm$ correspond to equal resp. opposite charges in the above-mentioned Bose gas interpretation). Assuming that the singularities of $\langle\tilde{\varphi}\tilde{\varphi}\rangle$ are only logarithmic, i.e.

$$\langle\tilde{\varphi}_t(x)\tilde{\varphi}_{t'}(x')\rangle = s(t, x; t', x') + r(t, x; t', x') \tag{28}$$

where

$$s(t, x; t', x') = -c_1\ln|x-x'| - c_2\ln|t-t'| = s_1(t, t') + s_2(x, x') \tag{29}$$

and r is a continuous function (such a behaviour is a consequence of the conditions (23) and (24); we return to this point in a subsequent publication) we can bound the integral (27) by an integral of a continuous function and the integral corresponding to

$$\langle\varphi_t(x)\varphi_{t'}(x')\rangle = s(t, x; t', x'). \tag{30}$$

Then, we can again separate the integral (27) into a first one corresponding to s_1 and a second one corresponding to s_2. Such integrals correspond to those one encounters in the study of a charged Bose gas. It has been shown in [24] that the nth-order integral is bounded by $(n/2)!^{\max(c\alpha^2, 1)} \sim n^{n/2}$ for $n\to\infty$. There is the $(n!)^{-1}$-factor coming from the time ordering. We need still to estimate the number of terms of the form (27). This number comes from the integration over W in (26). For a Gaussian integral we have

$$\langle|W|^n\rangle \sim (n/2)!n^{n/2} \qquad \text{as } n\to\infty.$$

So, we can conclude that for α and $\tilde{\lambda}$ small enough the series is convergent.

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